q-expansion of Hilbert modular forms

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Motivation

Where does q-expansion come from?

- Classical setting: $f : \mathcal{H} \to \mathbb{C}$, $f|_k\gamma = f$, holomorphic at cusps.
- Geometric setting: Katz's interpretation

Why is it important?

- Hecke eigenforms, Hecke eigenvalues
- Galois representations, Geometric Serre's conjecture

Modular forms $\rightsquigarrow$ Hilbert modular forms

Set up

- $F/\mathbb{Q}$ totally real number field of degree n.
- $\mathfrak{o}_F$ ring of integers
- $\Sigma = \mathrm{Hom}_{\mathbb{Q}-\mathrm{alg}}(F, \mathbb{C}) = \{\sigma_1, \cdots, \sigma_n\}$. Each σ_i factors through $\mathbb{R}$.
- $\mathcal{H}_F := \mathcal{H}^\Sigma \cong \mathcal{H}^n$.
- $\mathrm{GL}_2^+(F) \hookrightarrow \mathrm{GL}_2^+(\mathbb{R})^\Sigma$ acts on $\mathcal{H}_F$ by components.
- Weight $k = (k_\tau) \in \mathbb{Z}^\Sigma$, such that $\exists k_0$, $k_\tau \equiv k_0 \bmod 2$, $\forall \tau \in \Sigma$.
- Congurence subgroup $\Gamma \subset \mathrm{GL}_2^+(F)$ (e.g. $\mathrm{GL}_2^+(\mathfrak{o}_F)$)

Hilbert modular forms

Let $f : \mathcal{H}_F \to \mathbb{C}$, $k = (k_\tau) \in \mathbb{Z}^\Sigma$, and $\gamma \in \mathrm{GL}_2^+(F)$. Define

$$f|_k\gamma(z) := (\det\gamma)^{k+m-t} j(\gamma, z)^{-k} f(\gamma z),$$

where $m_\tau = (k_0 - k_\tau)/2$ and

$$(\det\gamma)^{k+m-t} j(\gamma, z)^{-k} = \prod_{\tau\in\Sigma} \left((\det\gamma_\tau)^{k_\tau + m_\tau - 1} (c_\tau z + d_\tau)^{-k_\tau} \right)$$

Definition (Hilbert modular forms)

A *Hilbert modular form* of *level* Γ and *weight* k is a holomorphic function $f : \mathcal{H}_F \to \mathbb{C}$ such that $f|_k\gamma = f$, for all $\gamma \in \Gamma$.

q-expansion

Let f be a Hilbert modular form of weight k and level $\mathrm{GL}_2^+(\mathfrak{o}_F)$. Then f has a Fourier expansion

$$f(z) = \sum_{\xi \in \mathfrak{d}^{-1}} a_\xi q^\xi,$$

where where $\mathfrak{d}$ is the different of F over $\mathbb{Q}$ and $q^\xi := e^{2\pi i \mathrm{tr}(\xi z)}$

q-expansion

Proposition (Koecher's principle)

Let f be a Hilbert modular form of weight k and level $\mathrm{GL}_2^+(\mathfrak{o}_F)$. If $[F:\mathbb{Q}] = n > 1$, then f is holomorphic at cusps in the following sense:

$$f(z) = a_0 + \sum_{\xi \in \mathfrak{d}_+^{-1}} a_\xi q^\xi$$

The above q-expansion in classical setting agrees with the q-expansion obtained in geometric setting.

Problem: Not practical to work with algorithmically.

q-expansion as multivariate formal power series

- M_k is the space of Hilbert modular forms, $f \in M_k$.
- Fix a $\mathbb{Z}$-basis $\{w_1, \cdots, w_n\}$ of $\mathfrak{d}^{-1}$.

$$f(z) = a_0 + \sum_{m=(m_i)\in\mathbb{Z}^n} a_m q_1^{m_1} \cdots q_n^{m_n}, \quad \text{with } q_i = e^{2\pi i \operatorname{tr}(w_i z)}$$

$$\bigoplus_k M_k \hookrightarrow \mathbb{C}[[q_1^{\pm}, \cdots, q_n^{\pm}]].$$

In fact, M_k can be embedded into the subring of multivariate formal power series.

Advantage: Practical in computation.

Computation in Magma

Let $\mathcal{S}_k$ be the space of adelic Hilbert cusp forms of weight k and $\mathbb{T}$ be its Hecke algebra.

- Magma computes the Hecke operators on $\mathcal{S}_k$, if $k_\tau \geq 2$, $\forall \tau$.
- We can obtain q-expansions by the duality theorem.

Theorem (Duality)

There is an isomorphism

$$\begin{aligned} \mathcal{S}_k &\longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{T}, \mathbb{C}) \\ f &\mapsto [T_{\mathfrak{a}} \mapsto a_1(T_{\mathfrak{a}} f)] \end{aligned}$$

Partial weight one case

Idea:
Let $E_{k'}$ be an Eisenstein series of parallel weight k'. Multiplying by $E_{k'}$ gives an injection

$$S_k \xrightarrow{\cdot E_{k'}} S_{k+k'}.$$

We are able to compute the q-expansion in this higher weight.

Main Application:
Weight part of geometric Serre's conjecture

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Thank you!